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S6, Mathematics for combination , Unit 1: Complex Numbers

LESSON 1: Algebraic form of a complex number

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Unit 1:Complex Numbers
1.0 Introduction

- You already know how to find the square root of a positive real number in sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$, but the problem occurs when finding square root of negative numbers.

Consider the following problem:

Solve in set of real number the following equations:

1. $x^2+6x+8=0$ 2. $x^2+4=0$

For equation 1), we have solution in $\mathbb{R}$ but equation 2) does not have solution in $\mathbb{R}$. Therefore, the set $\mathbb{R}$ is not sufficient to contain solutions of some equations.

There is a new set called **set of complex numbers**, in which the square root of negative number exists.

1.1. Concepts of complex numbers

A **complex number** is a number that can be put in the form $a+bi$, where a and b are real numbers and $i=\sqrt{-1}$ (i being the first letter of the word "imaginary").

The set of all complex numbers is denoted by $\mathbb{C}$ and is defined as $\mathbb{C}=\{z=a+bi : a,b\in\mathbb{R} \ \ and \ \ i^2=-1\}$

The real number a of the complex number $z=a+bi$ is called the **real part** of z and denoted by $Re(z)$ or $\Re(z)$; the real number b is called the **imaginary part** of z and denoted by $Im(z)$ or $\Im(z)$.

1.1. Concepts of complex numbers

- **Example 1.1**

Give two examples of complex numbers.

- **Solution**

There are several answers. For example $-3.5+2i$ and $4-6i$, where $i^2=-1$, are complex numbers.

- **Example 1.2**

Show the real part and imaginary part of the complex number $-3+4i$.

- **Solution**

$$\operatorname{Re}(-3+4i)=-3 \text{ and } \operatorname{Im}(-3+4i)=4$$

1.1. Concepts of complex numbers

Remarks

a) It is common to write a for $a+0i$ and bi for $0+bi$. Moreover, when the imaginary part is negative, it is common to write $a-bi$ with b> 0 instead of $a+(-b)i$, for example $3-4i$ instead of $3+(-4)i$.

b) A complex number whose real part is zero is said to be **purely imaginary** whereas a complex number whose imaginary part is **zero** is said to be a **real number** or simply **real**.

Therefore, all elements of $\mathbb{R}$ are elements of $\mathbb{C}$; and we can simply write $\mathbb{R}\subset\mathbb{C}$.

1.2. Algebraic form of a complex number

Recall that the set of all complex numbers is denoted by $\mathbb{C}$ and is defined as $\mathbb{C} = \{z = a + bi : a, b \in \mathbb{R} \ \ and \ \ i^2 = -1\}$.

$z = a + bi$ is the algebraic (or standard or Cartesian or rectangular) form of the complex number z.

- **1.2.1. Definition and properties of "*i*"**

For a complex number $z = a + bi$, i is called an imaginary unit.

Properties of imaginary unit i

The powers of imaginary unit are: $i^1 = i, \ \ i^2 = -1, \ i^3 = -i, \ i^4 = 1$.

If we continue, we return to the same results; the imaginary unit, i, "cycles" through 4 different values each time we multiply as it is illustrated in figure 1.1.

1.2. Algebraic form of a complex number

- 1.2.1. Definition and properties of “*i*”

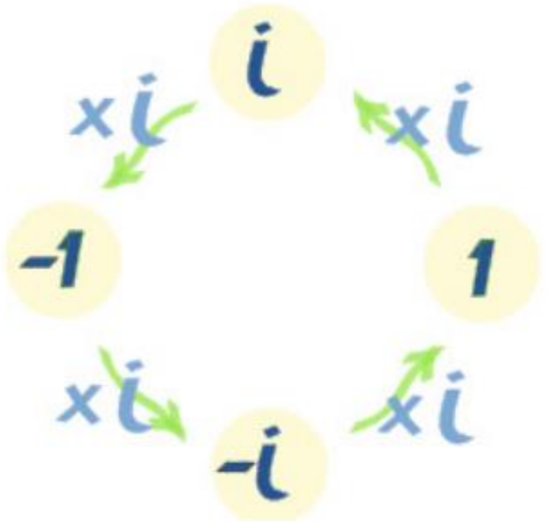


Figure 1.1. *Rotation of imaginary unit i*

Other exponents may be regarded as $4k+m$, $k=0,1,2,3,4,5,...$ and $m=0,1,2,3$.

Thus, the following relations may be used:

$i^{4k}=1,\ i^{4k+1}=i,\ i^{4k+2}=-1,\ i^{4k+3}=-i$

1.2. Algebraic form of a complex number

- ***Example 1.3***

Find the value of i^{48}, i^{801}, i^{142} and i^{22775}

- ***Solution***

$$i^{48} = i^{4\times 12} = 1, \qquad i^{801} = i^{4\times 200+1} = i,$$

$$i^{142} = i^{4\times 35+2} = -1, \qquad i^{22775} = i^{4\times 5693+3} = -i$$

- ***1.2.2. Geometric representation of complex numbers***

*A complex number can be visually represented as a pair of numbers (a, b) forming a vector from the origin or point on a diagram called **Argand** diagram (or **Argand** plane), named after Jean-Robert Argand, representing the complex plane. This plane is also called Gauss plane.*

*The x-axis is called the **real axis** and is denoted by **Re** while the y-axis is known as the **imaginary axis**; denoted **Im** as illustrated in fig.1.2.*

1.2. Algebraic form of a complex number

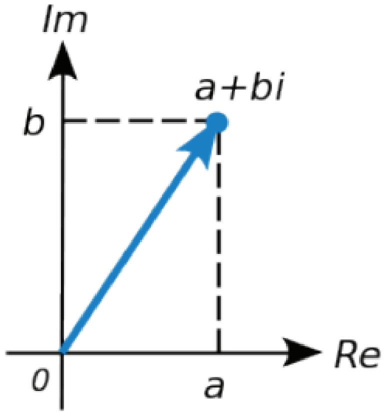


Figure 1.2. *Geometric representation of a complex number*

The Argand diagram fig 1.2 represents complex number $z = a + bi$ both as a point $P(\mathrm{a}, \mathrm{b})$ and as a vector $\overrightarrow{OP}$.

$z = (a, b)$ is a **geometric** form of the complex number z .

Notice

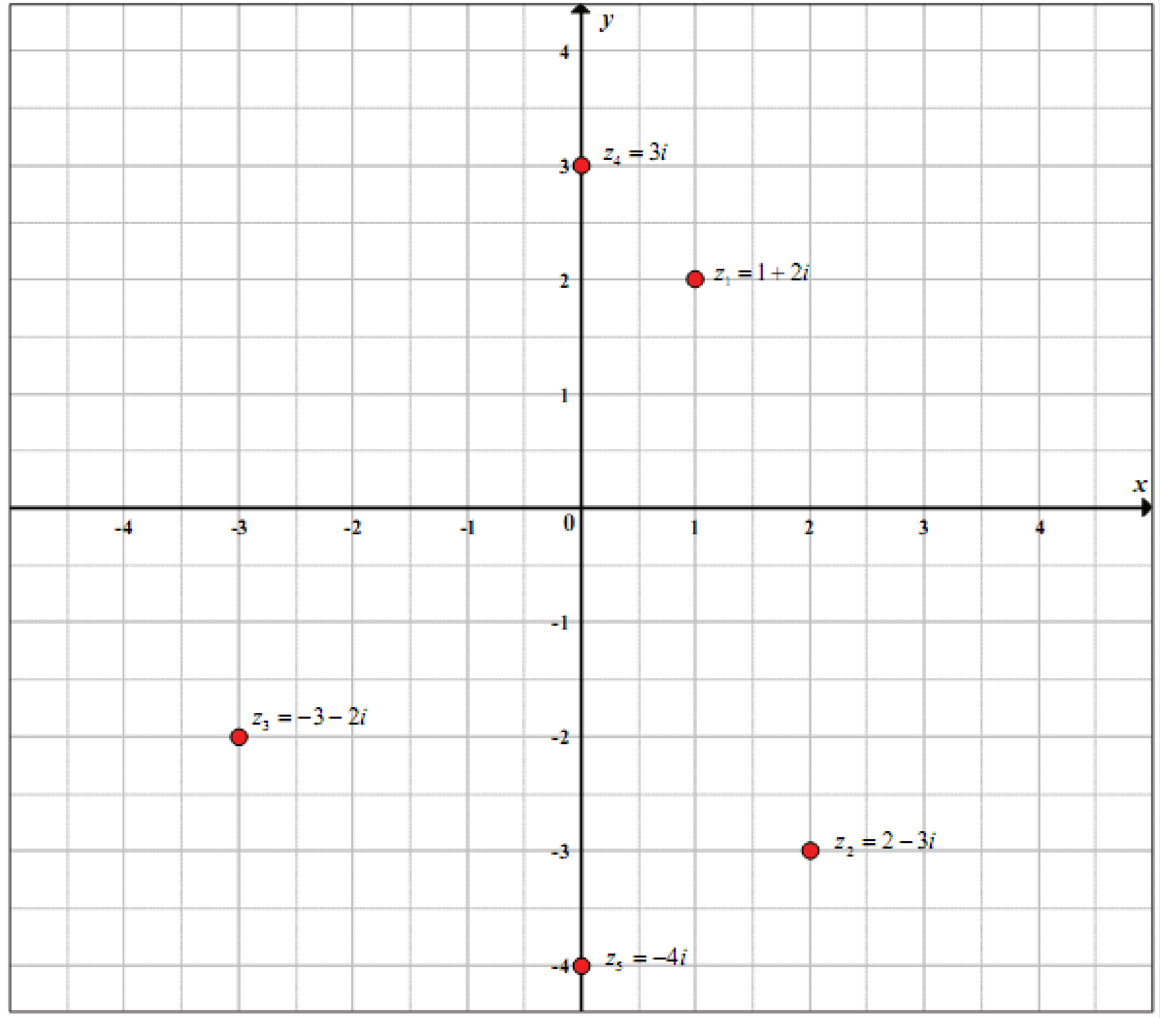
In complex plane, we will no longer talk about coordinates but affixes. The **affix** $z = a + bi$ of a point is plotted as a point and position vector on an Argand diagram; $a + bi$ is the **rectangular expression** of the point.

1.2. Algebraic form of a complex number

- **Example 1.4**

$z_1 = 1 + 2i,\ z_2 = 2 - 3i,\ z_3 = -3 - 2i,\ z_4 = 3i$ and $z_5 = -4i$.

- **Solution:**



1.2. Algebraic form of a complex number

- 1.2.3. Modulus of a complex number

The distance from origin to the point (x, y) corresponding to the complex number $z = x + yi$ is called the **modulus** (or **magnitude** or **absolute value**) of z and is denoted by $|z|$ or $|x+iy|$. Thus, modulus of z is given by $r = |z| = \sqrt{x^2 + y^2}$.

Example 1.5

Find the modulus of $4-3i$

Solution

$|4-3i| = \sqrt{16+9} = 5$

Example 1.6

Find the modulus of i

Solution

$|i| = \sqrt{0^2 + 1^2} = 1$

Example 1.7

Find the modulus of -3

Solution

$|-3| = \sqrt{(-3)^2 + 0^2} = 3$

Example 1.8

Find the modulus of $\frac{1}{2}\left(1+i\sqrt{3}\right)$

Solution

$$\left|\frac{1}{2}\left(1+i\sqrt{3}\right)\right| = \frac{1}{2}\left|1+i\sqrt{3}\right|$$
$$= \frac{1}{2}\sqrt{1+3} = 1$$

1.2. Algebraic form of a complex number

Properties of modulus

Let z, w be complex numbers different from 0, thus

a) $|z|^2 = [R_e(z)]^2 + [Im(z)]^2$

b) $|z|^2 = z\bar{z}$

c) $\mathrm{Re}(z) \le |\mathrm{Re}(z)| \le |z|$

d) $\mathrm{Im}(z) \le |\mathrm{Im}(z)| \le |z|$

e) $|zw| = |z||w|$

f) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

g) $|z+w| \le |z| + |w|$

i) $|z-w| \ge |z| - |w|$

Example 1.9

Find the modulus of $\frac{5}{3-4i}$

Solution

$$\left|\frac{5}{3-4i}\right| = \frac{|5|}{|3-4i|}$$

$$= \frac{5}{\sqrt{9+16}} = 1$$

Example 1.10

Find the modulus of $\frac{2+i}{1-3i}$

Solution

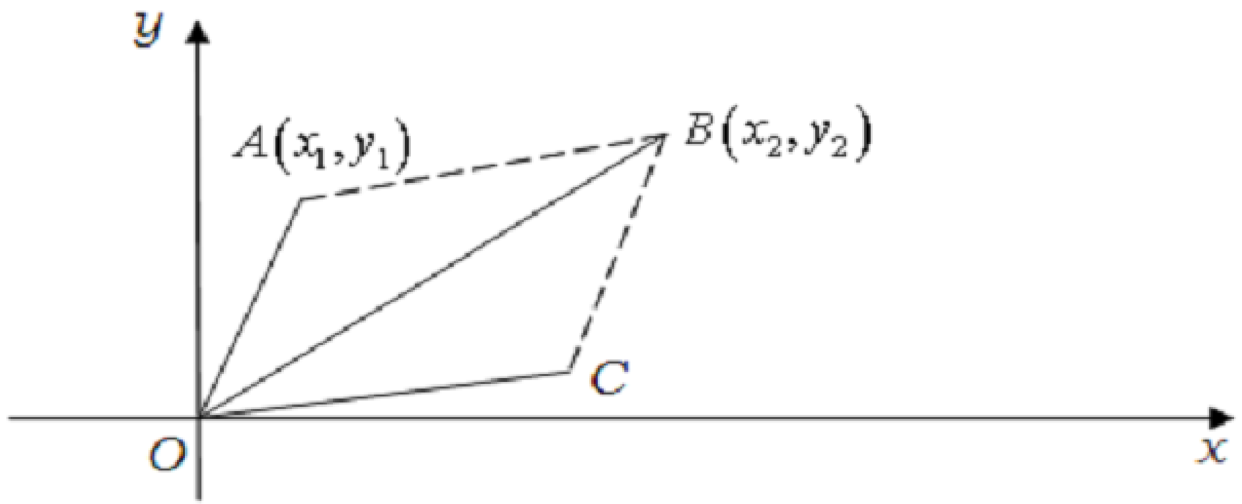
$$\left|\frac{2+i}{1-3i}\right| = \frac{|2+i|}{|1-3i|}$$

$$= \frac{\sqrt{5}}{\sqrt{10}} = \frac{\sqrt{2}}{2}$$

1.2. Algebraic form of a complex number

Interpretation of $|z_B - z_A|$

Consider the complex number $z_A = x_1 + iy_1$, where $z_A = x_1 + iy_1$ and $z_B = x_2 + iy_2$. The points A and B represent z_A and z_B respectively.



Then, $z = (x_2 - x_1) + i(y_2 - y_1)$ and is represented by the point C. This makes $OABC$ a parallelogram.

From this, it follows that $|z_B - z_A| = \overline{OC} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

That is to say, $|z_B - z_A|$ is the length AB in the Argand diagram.

1.2. Algebraic form of a complex number

If the complex number z_A is represented by the point A, and the complex number z_B is represented by the point B, then $|z_B - z_A| = \overline{AB}$,

Example 1.11

Let A and B be the points with affixes $z_A = -1 - i$, $z_B = 2 + 2i$. Find $\overline{AB}$.

Solution

$$\overline{AB} = |z_B - z_A| = |3 + 3i| = \sqrt{9+9} = 3\sqrt{2}$$

1.2. Algebraic form of a complex number

- **Loci related to the distances on Argand diagram**

A locus is a path traced out by a point subjected to certain restrictions. Paths can be traced out by points representing variable complex numbers on an Argand diagram just as they can in other coordinate systems.

Consider the simplest case first, when the point P represents the complex number z such that $|z| = R$. This means that the distance of OP from the origin O is constant and so P will trace out a circle.

$|z| = R$ represents a circle with centre at origin and radius R.

If instead $|z - z_1| = R$, where z_1 is a fixed complex number represented by point A on Argand diagram, then

$|z - z_1| = R$ represents a circle with centre z_1 and radius R.

1.2. Algebraic form of a complex number

$|z-z_1|=|z-z_2|$ represents a straight line which is the perpendicular bisector (mediator) of the line segment joining the points z_1 and z_2.

- **Example 1.12**

If $\left|\frac{z+2}{z}\right|=2$ and point P represents z in the Argand plane, show that P lies on a circle and find the centre and radius of this circle.

- **Solution**

Let $z=x+iy$ where $x,y\in\mathbb{R}$

Then $\left|\frac{z+2}{z}\right|=2$ $\quad\Rightarrow |z+2|=2|z|$

$\Rightarrow |x+iy+2|=2|x+iy|$ $\quad\Rightarrow |x+2+iy|=2|x+iy|$

$\Rightarrow \sqrt{(x+2)^2+y^2}=2\sqrt{x^2+y^2}$

$\Rightarrow (x+2)^2+y^2=4x^2+4y^2$ $\quad$ [*squaring both sides*]

$\Rightarrow x^2+4x+4+y^2=4x^2+4y^2$ $\quad\Rightarrow -3x^2-3y^2+4x=-4$

1.2. Algebraic form of a complex number

$\Rightarrow 3x^2+3y^2-4x=4$ which is the equation of a circle with centre at $\left(\frac{2}{3},0\right)$ and with radius of length $\frac{4}{3}$.

- **Example 1.13**

Determine, in complex plane, the locus M of affix z such that $|z-2i|=|z+2|$.

- **Solution**

Let $z=x+yi$, we have

$|x+yi-2i|=|x+yi+2|$

$\Rightarrow |x+i(y-2)|=|x+2+yi|$

$\Rightarrow \sqrt{x^2+(y-2)^2}=\sqrt{(x+2)^2+y^2}$

$\Rightarrow x^2+(y-2)^2=(x+2)^2+y^2$ $\quad$ [*squaring both sides*]

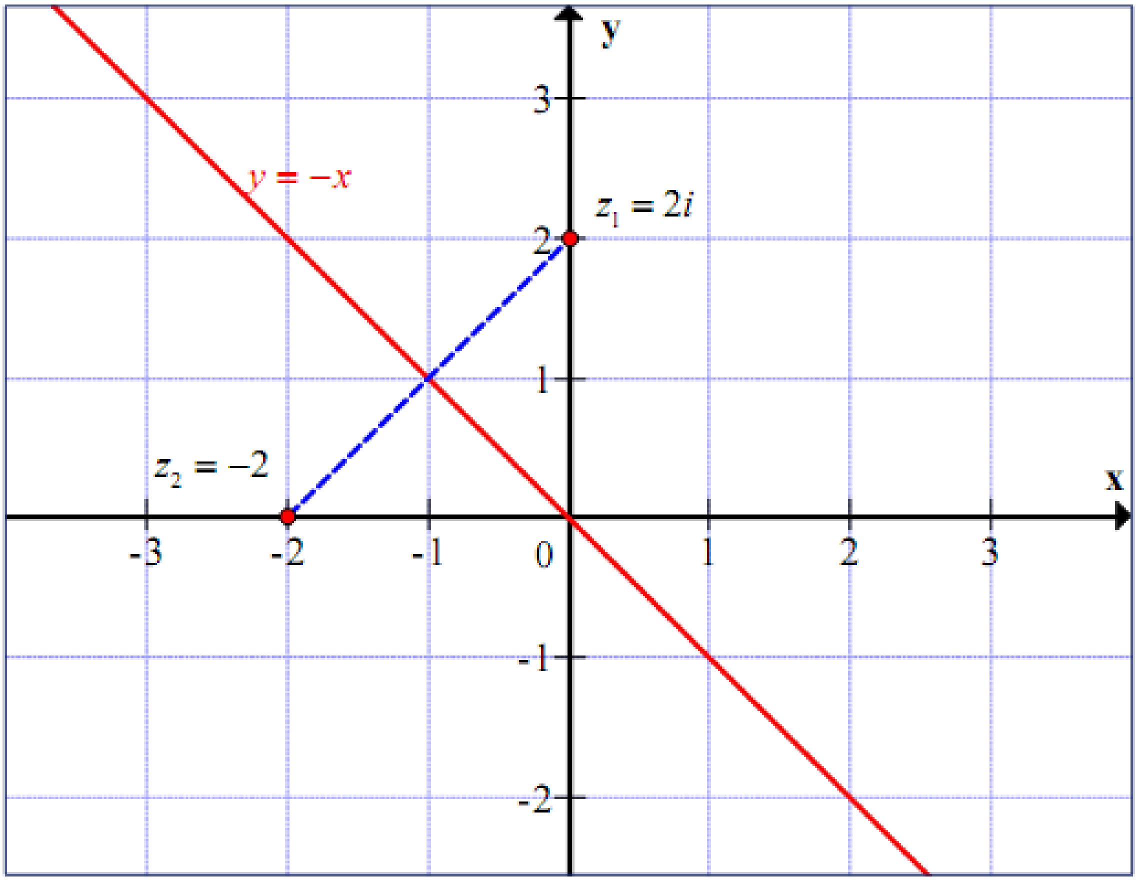
$\Rightarrow x^2+y^2-4y+4=x^2+4x+4+y^2$

$\Rightarrow -4y=4x$

$\Rightarrow y=-x$

1.2. Algebraic form of a complex number

This is a straight line, mediator of the line segment joining the points $z_1 = 2i$ and $z_2 = -2$. See the following figure.



1.2. Algebraic form of a complex number

- 1.2.4. Operations on complex number

Equality of two complex numbers

If two complex numbers, say $a+bi$ and $c+di$ are equal, then their real parts are equal and their imaginary parts equal. That is, $a+bi=c+di \Leftrightarrow a=c$ *and* $b=d$.

Example 1.14

Given $z=a+b-4i$, $w=2+bi$. Find the values of a and b if $z=w$.

Solution

$z=w \Leftrightarrow a+b=2, -4=b$

$a-4=2 \Rightarrow a=6$

Thus, $a=6$, $b=-4$

1.2. Algebraic form of a complex number

- **Addition and subtraction**

Consider the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ where $A(a,b), B(c,d)$ with $a,b,c,d \in \mathbb{R}$.

In fig. 1.3, $\overrightarrow{OX}$ is sum of the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

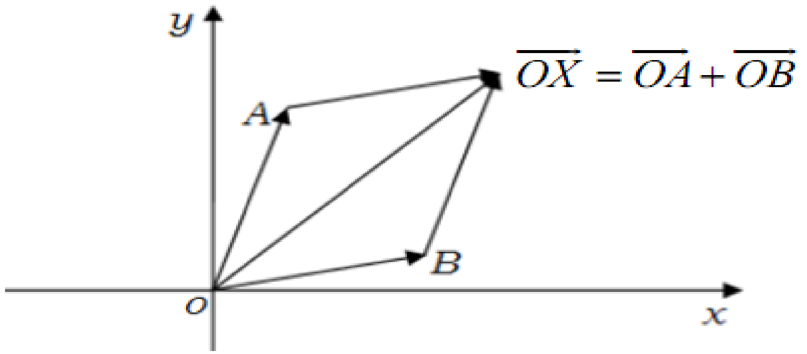


Figure 1.3. *Addition of two complex numbers*

Addition or **subtraction** of two complex numbers can be done geometrically by constructing a parallelogram (see Fig 1.3).

From activity 1.7, two complex numbers are added (or subtracted) by adding (or subtracting) separately the two real and the two imaginary parts. That is to say,

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

1.2. Algebraic form of a complex number

Particular element:

$$\left.\begin{array}{l}(a,b)+(0,0)=(a,b)\\(0,0)+(a,b)=(a,b)\end{array}\right\} \Rightarrow (0,0) \text{ is an additive identity.}$$

Example 1.15

Evaluate $z_1 + z_2$ if $z_1 = 3+4i$ and $z_2 = 1+2i$

Solution

$$(3+4i)+(1+2i)=(3+1)+(4+2)i=4+6i$$

Example 1.16

Evaluate $z_1 - z_2$ if $z_1 = 1-2i$ and $z_2 = 9+3i$

Solution

$$(1-2i)-(9+3i)=(1-9)+(-2-3)i=-8-5i$$

1.2. Algebraic form of a complex number

- Conjugate and opposite

The **complex conjugate** of the complex number $z = x + yi$, denoted by $\bar{z}$ or z^*, is obtained by changing the sign of the imaginary part. Hence, the complex conjugate of $z = x + yi$ is $\bar{z} = x - yi$.

The complex number $-z = -x - yi$ is the **opposite** of $z = x + yi$, symmetric of z with respect to 0.

Geometrical presentation of conjugate and opposite (negative) of complex number

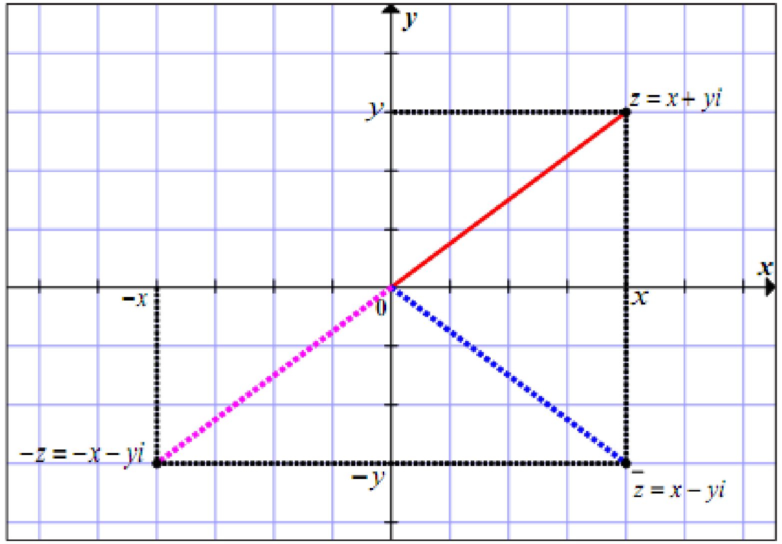


Figure 1.4. *Geometrical presentation of conjugate and opposite of a complex number*

1.2. Algebraic form of a complex number

- **Conjugate and opposite**

The **complex conjugate** of the complex number $z = x + yi$, denoted by $\bar{z}$ or z^*, is obtained by changing the sign of the imaginary part. Hence, the complex conjugate of $z = x + yi$ is $\bar{z} = x - yi$.

The complex number $-z = -x - yi$ is the **opposite** of $z = x + yi$, symmetric of z with respect to 0.

Geometrically, figure 1.4 shows that $\bar{z}$ is the "reflection" of z about the real axis while $-z$ is symmetric to z with respect to 0. In particular, conjugating twice gives the original complex number: $\bar{\bar{z}} = z$.

The real and imaginary parts of a complex number can be extracted using the conjugate:

$$\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z}) \qquad \mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$$

Moreover, a complex number is real if and only if it equals its conjugate.

1.2. Algebraic form of a complex number

- **Example 1.17**

Consider the complex number $z = 1 + 2i$. Show that

$$\operatorname{Re}(z) = \frac{1}{2}(z + \bar{z}) \text{ and } \operatorname{Im}(z) = \frac{1}{2i}(z - \bar{z})$$

- **Solution**

$$z = 1 + 2i, \qquad \bar{z} = 1 - 2i$$

$$\frac{1}{2}(z + \bar{z}) = \frac{1}{2}(1 + 2i + 1 - 2i) = \frac{1}{2}(2) = 1 = \operatorname{Re}(z)$$

$$\frac{1}{2i}(z - \bar{z}) = \frac{1}{2i}(1 + 2i - 1 + 2i) = \frac{1}{2i}(4i) = 2 = \operatorname{Im}(z)$$

Notice

- Conjugation distributes over the standard arithmetic operations:

  (i) $\overline{z + w} = \bar{z} + \bar{w}$ (ii) $\overline{zw} = \bar{z}\,\bar{w}$ (iii) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

  (iv) $\operatorname{Im}(z) = -Im(\bar{z})$ *and* $\operatorname{Re}(z) = \operatorname{Re}(\bar{z})$

- The complex number $-z = -x - yi$ is the opposite of $z = x+yi$, symmetric of z with respect to 0 and $z + |-z| = 0$.

1.2. Algebraic form of a complex number

Example 1.18

Find the conjugate of $z = u + w$, if $u = 6 + 2i$ and $w = 1 - 4i$.

Solution

Conjugate of $z = (6+2i)+(1-4i)$ is

$$\bar{z} = \overline{(6+2i)+(1-4i)} = \overline{7-2i} = 7+2i$$

Or

$$\bar{z} = \left(\overline{6+2i}\right)+\left(\overline{1-4i}\right) = (6-2i)+(1+4i) = 7+2i$$

Example 1.19

Find the conjugate of $z = u - w - t$, if $u = 2 - 3i$, $w = -1 - i$ and $t = 4 + 3i$.

Solution

Conjugate of $z = (2-3i)-(-1-i)-(4+3i)$ is

$$\bar{z} = \overline{(2-3i)-(-1-i)-(4+3i)} = \overline{-1-5i} = -1+5i$$

Or

$$\bar{z} = \overline{(2-3i)}-\overline{(-1-i)}-\overline{(4+3i)} = (2+3i)-(-1+i)-(4-3i) = -1+5i$$

1.2. Algebraic form of a complex number

- **Multiplication**

the multiplication of two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ is defined by the following formula:

$$z_1 \times z_2 = (a+bi)(c+di)$$
$$= (ac - bd) + (bc + ad)i$$

Alternatively, if $z_1(a,b), z_2(c,d)$ are complex numbers in geometric form, thus, $z_1 \cdot z_2 = (ac - bd, bc + ad)$.

In particular, the square of the imaginary unit is -1; since $i^2 = i \times i = -1$ or in geometric form $(0,1)(0,1) = (-1,0)$.

1.2. Algebraic form of a complex number

- **Inverse and division**

From activity 1.10, the **inverse** of $z=a+bi$ is given by

$\frac{1}{z}=z^{-1}=\frac{\overline{z}}{z\times\overline{z}}$ where $\overline{z}=a-bi$

Remark

The product $z\overline{z}=a^2+b^2$ is called the **norm** of $z=a+bi$ and is denoted by $\|z\|^2$ or $|z|^2$.

Thus,

$\frac{1}{z}=\frac{\overline{z}}{\|z\|^2}$ where $\overline{z}=a-bi$

Hence,

$$z^{-1}=\frac{\overline{z}}{a^2+b^2}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$$

If $z_1=a+bi$ and $z_2=c+di$, then

$$\frac{z_1}{z_2}=\frac{z_1\overline{z_2}}{\|z_2\|^2}$$

1.2. Algebraic form of a complex number

Or

$$\frac{z_1}{z_2}=\frac{a+bi}{c+di}=\left(\frac{ac+bd}{c^2+d^2}\right)+i\left(\frac{bc-ad}{c^2+d^2}\right)$$

Example 1.20

Find $\frac{1}{z}$ if $z=4+2i$

Solution

$$\frac{1}{z}=\frac{1}{4+2i}=\frac{4-2i}{4^2+2^2}=\frac{4}{20}-\frac{2}{20}i=\frac{1}{5}-\frac{1}{10}i$$

Example 1.21

Evaluate $\frac{-1+i}{i+2}$

Solution

$$\frac{-1+i}{i+2}=\frac{-1+i}{2+i}=\frac{(-1+i)(2-i)}{(2+i)(2-i)}=\frac{-2+1+2i+i}{1+4}=\frac{-1+3i}{5}.$$

1.2. Algebraic form of a complex number

Example 1.22

Find the real numbers x and y such that $(x+iy)(3-2i)=6-17i$.

Solution

$(x+iy)(3-2i)=6-17i$

$$\Rightarrow x+iy=\frac{6-17i}{3-2i}=\frac{(6-17i)(3+2i)}{(3-2i)(3+2i)}=\frac{52-39i}{13}=4-3i$$

Thus, $x=4$ and $y=-3$

Alternative method

$(x+iy)(3-2i)=6-17i$

$$3x-2ix+3iy+2y=6-17i \Leftrightarrow 3x+2y+(-2x+3y)i=6-17i$$

$$\Leftrightarrow \begin{cases} 3x+2y=6 \\ -2x+3y=-17 \end{cases}$$

Solving this system, we get;

$x=4$ and $y=-3$

1.2. Algebraic form of a complex number

Remarks

1. Three distinct points A, B and C with affixes z_1, z_2 and z_3 respectively are collinear if and only if $\dfrac{z_C - z_A}{z_B - z_A} \in IR$
2. The non-zero vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular if and only if $\dfrac{z_C - z_A}{z_B - z_A}$ is pure imaginary different from zero.

Example 1.23

Let A, B and C be the points with affixes $z_A = -1 - i$, $z_B = 2 + 2i$ and $z_C = 3 + 3i$ respectively. Show that they are collinear points.

Solution

$$\frac{z_C - z_A}{z_B - z_A} = \frac{(3+3i)-(-1-i)}{(2+2i)-(-1-i)}$$

$$\frac{4+4i}{3+3i} = \frac{4(1+i)}{3(1+i)} = \frac{4}{3} \in IR$$

Thus, z_A, z_B and z_C are collinear.

1.2. Algebraic form of a complex number

Example 1.24

Let A, B and C be the points with affixes $z_A = 2+i$, $z_B = 3+2i$ and $z_C = 1+2i$ respectively.

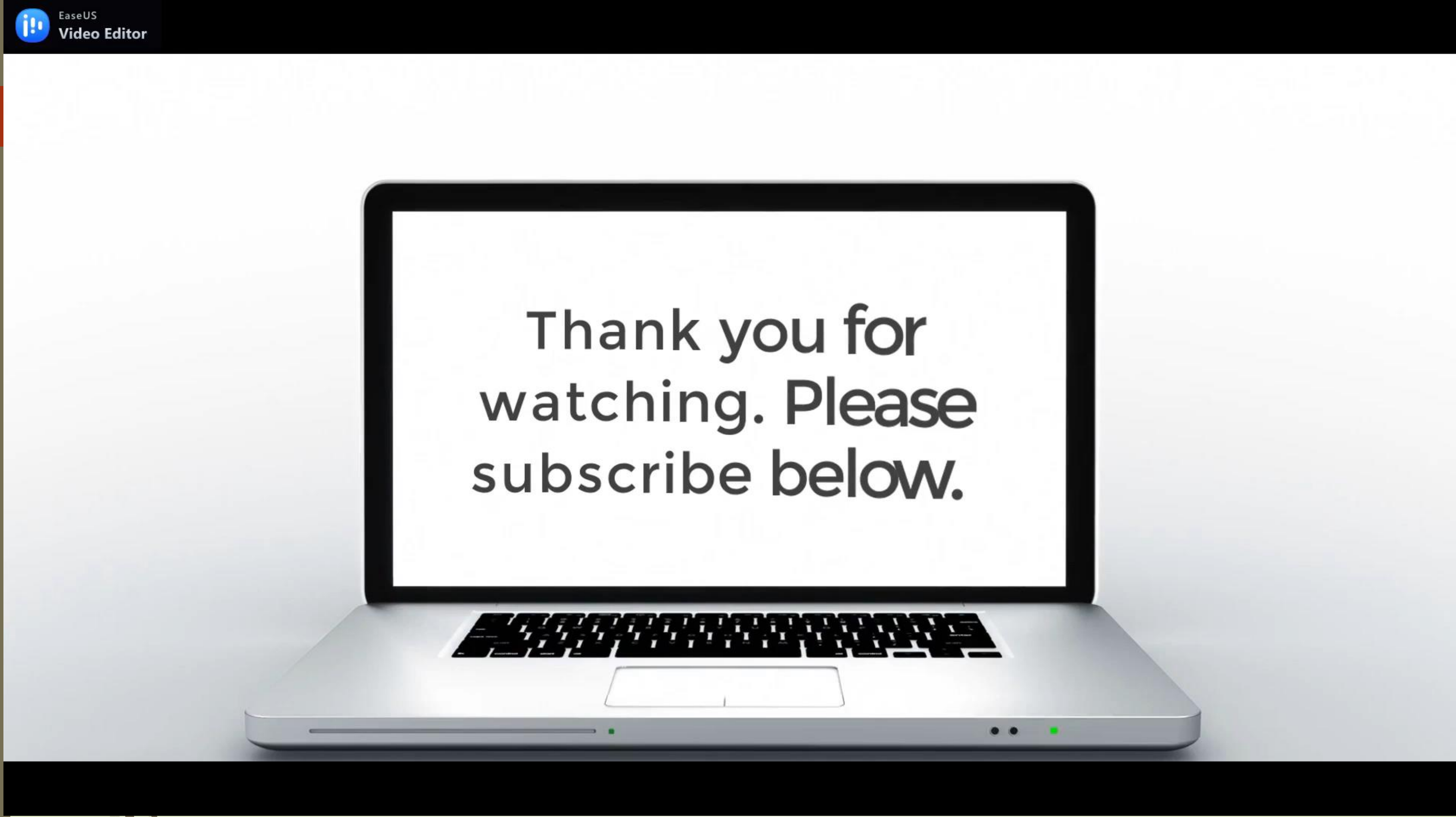
Show that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular.

Solution

$z_C - z_A = \overrightarrow{AC} = -1+i \qquad z_B - z_A = \overrightarrow{AB} = 1+i$

$$\frac{z_C - z_A}{z_B - z_A} = \frac{-1+i}{1+i} = \frac{(-1+i)(1-i)}{\sqrt{2}} = i\sqrt{2}$$

This is pure imaginary different from zero. Thus, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular.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